

MacMahon Master Theorem

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Reference: [1], page 118, exercise 19.

Theorem 1. The Dixon's identity is: for $a, b, c \in \mathbb{N}$,

$$\sum_k (-1)^k \binom{a+b}{a+k} \binom{b+c}{b+k} \binom{c+a}{c+k} = \frac{(a+b+c)!}{a!b!c!} = \binom{a+b+c}{a, b, c}.$$

Proof.

□

- Question 1** (Ning Jiang). 1) Find the constant coefficient in the expansion of $(1 - \frac{x}{y})^m (1 - \frac{y}{x})^n$.
 2) Find the constant coefficient in the expansion of $\prod_{1 \leq i \neq j \leq n} (1 - \frac{x_i}{x_j})$.
 3) Find the constant coefficient in the expansion of $\prod_{1 \leq i \neq j \leq n} (1 - \frac{x_i}{x_j})^{d_j}$, $d_j \in \mathbb{N}$.

Solution. 1) $\sum_k \binom{m}{k} \binom{n}{k} = \sum_k \binom{m}{k} \binom{n}{n-k} = \binom{m+n}{n}$.
 2) $n!$.

3) In the Dixon's identity, when $a = b = c$, the equality becomes $\sum_k (-1)^k \binom{2a}{a+k}^3 = \frac{(3a)!}{(a!)^3}$. When $n = 3, d_1 = a, d_2 = b, d_3 = c$, $(-1)^k \sum_i \binom{b}{i} \binom{a}{i-k} = (-1)^k \sum_i \binom{b}{i} \binom{a}{a+k-i} = (-1)^k \binom{a+b}{a+k}$ is the coefficient of $(\frac{x_1}{x_2})^k$ in the expansion of $(1 - \frac{x_1}{x_2})^b (1 - \frac{x_2}{x_1})^a$. Similarly we may list 3 equalities below:

$$\text{coeff} < (\frac{x_1}{x_2})^k, (1 - \frac{x_1}{x_2})^b (1 - \frac{x_2}{x_1})^a > = (-1)^k \binom{a+b}{a+k},$$

$$\text{coeff} < (\frac{x_2}{x_3})^k, (1 - \frac{x_2}{x_3})^c (1 - \frac{x_3}{x_2})^b > = (-1)^k \binom{b+c}{b+k},$$

$$\text{coeff} < (\frac{x_3}{x_1})^k, (1 - \frac{x_3}{x_1})^a (1 - \frac{x_1}{x_3})^c > = (-1)^k \binom{c+a}{c+k},$$

Putting the above equalities together, we conclude that when $n = 3$,

$$\text{coeff} < 1, \prod_{1 \leq i \neq j \leq 3} (1 - \frac{x_i}{x_j})^{d_j} > = \frac{(d_1 + d_2 + d_3)!}{d_1!d_2!d_3!} = \binom{d_1 + d_2 + d_3}{d_1, d_2, d_3},$$

We want to imitate the proof of the MacMahon Master Theorem. Define

$$\begin{aligned} G(d_1, \dots, d_n) &= \text{coeff} < x_1^{(n-1)d_1} \dots x_n^{(n-1)d_n}, \prod_{1 \leq i \neq j \leq n} (x_j - x_i)^{d_j} > \\ &= \text{coeff} < 1, \prod_{1 \leq i \neq j \leq n} (1 - \frac{x_i}{x_j})^{d_j} >. \end{aligned}$$

Let t_1, \dots, t_n be another set of formal variables, then

$$\begin{aligned}
F &= \sum_{d_1, \dots, d_n \geq 0} G(d_1, \dots, d_n) t_1^{d_1} \dots t_n^{d_n} = \sum_{d_1, \dots, d_n \geq 0} \text{coeff} < 1, \prod_{1 \leq i \neq j \leq n} (1 - \frac{x_i}{x_j})^{d_j} > t_1^{d_1} \dots t_n^{d_n} \\
&= \text{coeff} < 1, \sum_{d_1, \dots, d_n \geq 0} (t_j \prod_{1 \leq i \neq j \leq n} (1 - \frac{x_i}{x_j})^{d_j}) > = \text{coeff} < 1, \prod_{1 \leq j \leq n} (1 - t_j \prod_{i \neq j} (1 - \frac{x_i}{x_j}))^{-1} >, \\
g(x_1, \dots, x_n) &= \prod_{1 \leq j \leq n} (1 - t_j \prod_{i \neq j} (1 - \frac{x_i}{x_j})), \quad x_i = e^{2\pi J \theta_i}, \quad J = \sqrt{-1}, \\
1 - \frac{x_i}{x_j} &= 1 - e^{2\pi J(\theta_i - \theta_j)} = -2J \sin \pi(\theta_i - \theta_j) e^{\pi(\theta_i - \theta_j)}, \\
\prod_{i \neq j} (1 - \frac{x_i}{x_j}) &= (-2J)^{n-1} \prod_{i \neq j} \sin \pi(\theta_i - \theta_j) e^{\pi(\theta_i - \theta_j)} = (-2J)^{n-1} e^{\pi(\theta - n\theta_j)} \prod_{i \neq j} \sin \pi(\theta_i - \theta_j),
\end{aligned}$$

where $\theta = \sum_{i=1}^n \theta_i$. Qualitatively, g^{-1} is a rational function of t_1, \dots, t_n and x_1, \dots, x_n , so the constant coefficient that we concern is a rational function of t_1, \dots, t_n , and it has a power series expansion near $(t_1, \dots, t_n) = (0, \dots, 0)$. Moreover, we know in advance that $G(d_1, \dots, d_n) = \binom{d_1 + \dots + d_n}{d_1, \dots, d_n}$, so it suffices to show that

$$\begin{aligned}
F &= \sum_{d_1, \dots, d_n \geq 0} \binom{d_1 + \dots + d_n}{d_1, \dots, d_n} t_1^{d_1} \dots t_n^{d_n} = \frac{1}{1 - (t_1 + \dots + t_n)}, \\
g(x_1, \dots, x_n) &= \prod_{1 \leq j \leq n} (1 - t_j (-2J)^{n-1} e^{\pi(\theta - n\theta_j)} \prod_{i \neq j} \sin \pi(\theta_i - \theta_j)),
\end{aligned}$$

□

Theorem 2 (MacMahon Master Theorem). Let $A = (a_{i,j})_{m \times m}$ be a complex matrix, and let x_1, \dots, x_m be formal variables. Consider the coefficient

$$G(k_1, \dots, k_m) = \text{coeff} < x_1^{k_1} \dots x_m^{k_m}, \prod_{1 \leq i \leq m} (a_{i,1}x_1 + \dots + a_{i,m}x_m)^{k_i} >$$

Let t_1, \dots, t_m be another set of formal variables, and let $T = (\delta_{i,j}t_i)_{m \times m}$ be a diagonal matrix, then

$$\sum_{k_1, \dots, k_m \geq 0} G(k_1, \dots, k_m) t_1^{k_1} \dots t_m^{k_m} = \frac{1}{\det(I_m - TA)},$$

Proof.

$$\begin{aligned}
&\text{coeff} < x_1^{k_1} \dots x_m^{k_m}, \prod_{1 \leq i \leq m} (a_{i,1}x_1 + \dots + a_{i,m}x_m)^{k_i} > \\
&= \text{coeff} < 1, \prod_{1 \leq i \leq m} \left(\frac{a_{i,1}x_1 + \dots + a_{i,m}x_m}{x_i} \right)^{k_i} >
\end{aligned}$$

Denote $LHS = \sum_{k_1, \dots, k_m \geq 0} G(k_1, \dots, k_m) t_1^{k_1} \dots t_m^{k_m}$, then

$$\begin{aligned}
LHS &= \sum_{k_1, \dots, k_m \geq 0} \text{coeff} < 1, \prod_{1 \leq i \leq m} \left(\frac{a_{i,1}x_1 + \dots + a_{i,m}x_m}{x_i} \right)^{k_i} > t_1^{k_1} \dots t_m^{k_m} \\
&= \text{coeff} < 1, \sum_{k_1, \dots, k_m \geq 0} \prod_{1 \leq i \leq m} \left(\frac{a_{i,1}x_1 + \dots + a_{i,m}x_m}{x_i} \right)^{k_i} t_i^{k_i} > \\
&= \text{coeff} < 1, \prod_{1 \leq i \leq m} (1 - \left(\frac{a_{i,1}x_1 + \dots + a_{i,m}x_m}{x_i} \right) t_i)^{-1} >
\end{aligned}$$

Fix t_1, \dots, t_m with small norm, let

$$g(x_1, \dots, x_m) = \prod_{1 \leq i \leq m} \left(1 - \left(\frac{a_{i,1}x_1 + \dots + a_{i,m}x_m}{x_i}\right)t_i\right),$$

Let $x = (x_1, \dots, x_m)^{\text{tr}}$ be a column vector. Then we may express g as

$$g(x_1, \dots, x_m) = \prod_{1 \leq i \leq m} \left(1 - \frac{(TAx)_i}{x_i}\right) = \prod_{1 \leq i \leq m} \frac{(Lx)_i}{x_i},$$

Here $L = I_m - TA$ is an invertible matrix since $|T|$ is small. Take $x_i = e^{2\pi J\theta_i}$, $J = \sqrt{-1}$, we have $dx_i = 2\pi Jx_i d\theta_i$. Moreover, let $Lx = y$ and take $y_i = r_i e^{2\pi J\theta'_i}$, $r_i > 0$, $dy_i = 2\pi Jy_i d\theta'_i$,

$$\begin{aligned} LHS &= \int_{\mathbb{T}^m} \frac{1}{g} d\theta_1 \dots d\theta_m = \int_{\mathbb{T}^m} \prod_{1 \leq i \leq m} \frac{x_i}{(Lx)_i} d\theta_1 \dots d\theta_m \\ &= \int_{\mathbb{T}^m} \prod_{1 \leq i \leq m} \frac{1}{2\pi J} \frac{dx_i}{(Lx)_i} = \frac{1}{(2\pi J)^m} \int_{L\mathbb{T}^m} \prod_{1 \leq i \leq m} \frac{d(L^{-1}y)_i}{y_i} \\ &= \frac{1}{(2\pi J)^m} \int_{L\mathbb{T}^m} \det(L^{-1}) \frac{dy_1 \dots dy_m}{y_1 \dots y_m} \\ &= \det(L^{-1}) \int_{\mathbb{T}^m} d\theta'_1 \dots d\theta'_m = \det(L^{-1}) = \frac{1}{\det(I_m - TA)}, \end{aligned}$$

□

We may also calculate other coefficients. Let $l_1, \dots, l_m \in \mathbb{Z}$, $l_1 + \dots + l_m = 0$, consider the coefficient

$$G(k_1, \dots, k_m) = \text{coeff} \langle x_1^{l_1+k_1} \dots x_m^{l_m+k_m}, \prod_{1 \leq i \leq m} (a_{i,1}x_1 + \dots + a_{i,m}x_m)^{k_i} \rangle,$$

with corresponding generating function

$$F = \sum_{k_1, \dots, k_m \geq 0} G(k_1, \dots, k_m) t_1^{k_1} \dots t_m^{k_m},$$

then

$$\begin{aligned} F &= \sum_{k_1, \dots, k_m \geq 0} \text{coeff} \langle x_1^{l_1} \dots x_m^{l_m}, \prod_{1 \leq i \leq m} \left(\frac{a_{i,1}x_1 + \dots + a_{i,m}x_m}{x_i}\right)^{k_i} \rangle t_1^{k_1} \dots t_m^{k_m} \\ &= \text{coeff} \langle x_1^{l_1} \dots x_m^{l_m}, \prod_{1 \leq i \leq m} \left(1 - \left(\frac{a_{i,1}x_1 + \dots + a_{i,m}x_m}{x_i}\right)t_i\right)^{-1} \rangle, \end{aligned}$$

Theorem 3 (Feuerbach's theorem). The nine-point circle of a triangle ABC is tangent to its inner inscribed circle.

Proof. It suffices to show that $O_1I = \frac{R}{2} - r$.

$$IO_1 = IO + \frac{1}{2}OH, \quad OH = OA + OB + OC, \quad OI = \frac{a}{a+b+c}OA + \frac{b}{a+b+c}OB + \frac{c}{a+b+c}OC,$$

$$IO_1 = \frac{1}{a+b+c}(xOA + yOB + zOC), \quad x = p - a, \quad y = p - b, \quad z = p - c, \quad p = \frac{a+b+c}{2},$$

$$|IO_1|^2 = \frac{R^2}{(a+b+c)^2}(x^2 + y^2 + z^2 + 2xy(1 - 2\sin^2 C) + 2yz(1 - 2\sin^2 A) + 2xz(1 - 2\sin^2 B))$$

$$xy = (p-a)(p-b) = \frac{c^2 - (a-b)^2}{4} = R^2(\sin^2 C - (\sin A - \sin B)^2)$$

$$1 = p^2 - 4xy\sin^2 C - 4yz\sin^2 A - 4xz\sin^2 B = p^2 - 4R^2(\sin^2 C - (\sin A - \sin B)^2)\sin^2 C - 4R^2(\sin^2 A - (\sin B - \sin C)^2)\sin^2 A - 4R^2(\sin^2 B - (\sin C - \sin A)^2)\sin^2 B,$$

$$1 - p^2 = -4R^2(\sin^4 A + \sin^4 B + \sin^4 C) + 8R^2(\sin^2 A\sin^2 B + \sin^2 B\sin^2 C + \sin^2 C\sin^2 A) - 8R^2\sin A\sin B\sin C(\sin A + \sin B + \sin C),$$

$$r = \frac{4R^2\sin A\sin B\sin C}{a+b+c}, \quad \left(\frac{R}{2} - r\right)^2 = \frac{R^2}{(a+b+c)^2}(p - 4R\sin A\sin B\sin C)^2$$

$$2 = p^2 - p \cdot 8R\sin A\sin B\sin C + 16R^2\sin^2 A\sin^2 B\sin^2 C, \quad p = R(\sin A + \sin B + \sin C),$$

$$2 - p^2 = -8R^2(\sin A + \sin B + \sin C)\sin A\sin B\sin C + 16R^2\sin^2 A\sin^2 B\sin^2 C,$$

$$2(\sin^2 A\sin^2 B + \sin^2 B\sin^2 C + \sin^2 C\sin^2 A) - \sin^4 A - \sin^4 B - \sin^4 C - 4\sin^2 A\sin^2 B\sin^2 C$$

$$= \sin^2 A(1 - 2\sin^2 B)(1 - 2\sin^2 C) + \sin^2 A + 2\sin^2 B\sin^2 C - \sin^4 A - \sin^4 B - \sin^4 C$$

$$= -\sin^2 A\cos 2B\cos 2C + \sin^2 A\cos^2 A - (\sin^2 B - \sin^2 C)^2, \quad \sin^2 B - \sin^2 C = \sin A\sin(B-C),$$

$$-\cos 2B\cos 2C + \cos^2 A - \sin^2(B-C) = \frac{1}{2}\cos 2A - \frac{1}{2}\cos 2(B-C) + \frac{1 + \cos 2A}{2} - \frac{1 - \cos 2(B-C)}{2} = 0.$$

□

Theorem 4 (Casey's theorem). The circumcenter of $\triangle ABC$ is O , there's another circle centered at O_1 . t_A, t_B, t_C are lengths of tangent segments from A, B, C to circle O_1 . Then circle O_1 and circle O are inscribed to each other is equivalent to $at_A + bt_B = ct_C$, where $a = BC, b = AC, c = AB$.

Proof. 1) If circle O_1 and circle O are inscribed to each other, then

2) If $at_A + bt_B = ct_C$, then

$$c^2 t_C^2 - a^2 t_A^2 - b^2 t_B^2 = 2abt_A t_B,$$

$$LHS = c^2(t_{C0}^2 + r_0^2 - r^2) - a^2(t_{A0}^2 + r_0^2 - r^2) - b^2(t_{B0}^2 + r_0^2 - r^2) = LHS_0 + (r_0^2 - r^2)(c^2 - a^2 - b^2),$$

$$RHS^2 = 4a^2 b^2 t_A^2 t_B^2 = 4a^2 b^2 (t_{A0}^2 + r_0^2 - r^2)(t_{B0}^2 + r_0^2 - r^2) = 4a^2 b^2 (t_{A0}^2 t_{B0}^2 + (r_0^2 - r^2)(t_{A0}^2 + t_{B0}^2) + (r_0^2 - r^2)^2),$$

$$LHS^2 = LHS_0^2 + 2LHS_0(r_0^2 - r^2)(c^2 - a^2 - b^2) + (r_0^2 - r^2)^2(c^2 - a^2 - b^2)^2, \quad c^2 - a^2 - b^2 = -2ab\cos C,$$

$$RHS^2 - LHS^2 = (r_0^2 - r^2)4a^2 b^2 (t_{A0}^2 + t_{B0}^2 + 2t_{A0}t_{B0}\cos C) + (r_0^2 - r^2)^2 4a^2 b^2 \sin^2 C,$$

$$t_{A0} = \sqrt{\frac{l}{R}}AD, \quad t_{B0} = \sqrt{\frac{l}{R}}BD, \quad \cos C = -\cos D, \quad t_{A0}^2 + t_{B0}^2 + 2t_{A0}t_{B0}\cos C = \frac{l}{R}c^2,$$

$$RHS^2 - LHS^2 = (r_0^2 - r^2)\frac{4a^2 b^2 c^2 l}{R} + (r_0^2 - r^2)^2 \frac{a^2 b^2 c^2}{R^2} = (r_0^2 - r^2)\frac{a^2 b^2 c^2}{R^2}(4Rl + (r_0^2 - r^2)),$$

$$r^2 = r_0^2, \text{ or } r^2 = r_0^2 + 4Rl = (R-l)^2 + 4Rl = (R+l)^2$$

In both of the above two cases, circle O_1 and circle O are inscribed to each other. But actually if $r = R + l$, then there are no tangent segments from A, B, C to circle O_1 . □

Question 2 (2013 China TST p14). Suppose $\angle API = \alpha$, since $\angle AEF = \angle APE$, we have

$$\tan \alpha = \tan \angle AEF = \frac{EA \times FA}{AE \cdot FE},$$

$$AE \cdot FE = AP \sin \alpha DF - (DP - AP \cos \alpha - DQ)DQ, \quad DF = AI \cdot \frac{DP}{AP},$$

$$EA \times FA = (DP - DQ) \sin \alpha FA, \quad FA = \frac{DP}{\cos \alpha} - AP,$$

$$\begin{aligned} \frac{EA \times FA}{\tan \alpha} &= (DP - DQ) \sin \alpha \left(\frac{DP}{\cos \alpha} - AP \right) \frac{\cos \alpha}{\sin \alpha} = (DP - DQ)(DP - AP \cos \alpha) \\ &= AI \cdot DP \sin \alpha - (DP - AP \cos \alpha - DQ)DQ, \end{aligned}$$

$$DP(DP - AP \cos \alpha) = AI \cdot DP \sin \alpha + DQ^2, \quad DP^2 - DQ^2 = AI \cdot DP \sin \alpha + AP \cdot DP \cos \alpha = DP \cdot PI,$$

$$LHS = BP^2 - BQ^2 = 4R^2 \left(\cos^2 \frac{A}{2} - \sin^2 \frac{A}{2} \right) = 4R^2 \cos A, \quad \frac{DP}{IP} = \frac{PM}{PM - r} = \frac{2R \cos^2 \frac{A}{2}}{2R \cos^2 \frac{A}{2} - r},$$

$$\begin{aligned} PI^2 &= AI^2 + AP^2 = 4R^2 \left(\left(\cos \frac{C-B}{2} - \sin \frac{A}{2} \right)^2 + \sin^2 \frac{C-B}{2} \right) \\ &= 4R^2 \left(1 + \sin^2 \frac{A}{2} - \cos B - \cos C \right) = 4R^2 \cos A \frac{1 + \cos A - \frac{r}{R}}{1 + \cos A} \end{aligned}$$

$$\frac{r}{R} = \frac{r}{AI} \frac{AI}{R} = \sin \frac{A}{2} \cdot 2 \left(\cos \frac{C-B}{2} - \sin \frac{A}{2} \right) = \cos B + \cos C - 1 + \cos A$$

$$\left(1 + \cos A - \frac{r}{R} \right) \cos A = \left(1 + \sin^2 \frac{A}{2} - \cos B - \cos C \right) (1 + \cos A),$$

$$(2 - \cos B - \cos C) \cos A = (1 + \cos A) \left(1 + \sin^2 \frac{A}{2} - \cos A - \cos B - \cos C \right),$$

$$2 \cos A = (1 + \cos A) \left(1 + \sin^2 \frac{A}{2} \right) - \cos B - \cos C = 1 + \cos A + \sin^2 \frac{A}{2} (1 + \cos A) - \cos B - \cos C,$$

$$\frac{r}{R} = \sin^2 \frac{A}{2} (1 + \cos A) = 2 \sin^2 \frac{A}{2} \cos^2 \frac{A}{2} = \frac{\sin^2 A}{2}, \quad \sin^2 A = \frac{2r}{R},$$

Question 3 (2021 China TST p20). Let $Q = MN \cap FI$, $RFIA, MQIA, RFQM$ are cocyclic, $LI \perp NM$, it suffices to show that $\angle MLI = \angle KNM$.

$$\angle LIA = C + \frac{A}{2} = \angle NRD, \quad \angle LAI = \angle NDR, \quad \triangle LIA \sim \triangle NRD, \quad LI = AI \cdot \frac{RN}{RD},$$

Let $P = AE \cap \odot O$, then since $\angle NRI = \angle FAI = \angle EAI$, RIP are colinear. Coincidence: AQE are colinear. Let I' be the incenter of $\triangle DBC$, $Q' = II' \cap AE$, it suffices to show that $Q = Q'$, i.e.,

$$I'Q' = \frac{1}{2} I'I = \frac{b-c}{2}, \quad \frac{I'Q'}{r} = \frac{p-c-c \cos B}{c \sin B} = \frac{b-c}{2r}, \quad r = \frac{ac \sin B}{a+b+c}$$

$$\frac{2a(p-c-c \cos B)}{a+b+c} = \frac{(a+b-c)a - (a^2 + c^2 - b^2)}{a+b+c} = b-c, \quad 2r(p-c-c \cos B) = (b-c)c \sin B,$$

$$\frac{RN}{RD} = \frac{\sin \angle NPI}{\sin \angle DPI} = \frac{\sin \angle INP \cdot IN}{\sin \angle IDP \cdot ID}, \quad IK = AI \cdot \frac{\sin \angle FAI}{\sin \angle AKI},$$

$$\angle AKI = \angle AID - \angle EAI = B - C + \angle I'AN - \angle EAI = B - C + \angle I'AE,$$

$$\tan \angle KNM = \frac{IQ + IK \cos \alpha}{NQ - IK \sin \alpha}, \quad \alpha = \angle IDA, \quad \tan \angle MLI = \frac{MQ}{QI + LI},$$

$$IQ = \frac{b-c}{2}, \quad NQ = 2R \sin \frac{A}{2} \cos \frac{B-C}{2}, \quad MQ = 2R - NQ,$$

$$\beta = \angle FAD = \angle PDA, \quad AP = 2R \sin \beta, \quad \angle FAI = \beta - C - \frac{A}{2}, \quad \angle AKI = \pi - \alpha - \beta,$$

Assume $Y = AF \cap BC$, then

$$IN \sin \angle INP = AP \frac{IN}{2R} = AP \sin \frac{A}{2}, \quad ID \sin \angle IDP = d(I', AF) = EY \sin \beta + r \cos \beta,$$

It suffices to show that

$$(2R - NQ)(NQ - IK \sin \alpha) = (IQ + LI)(IQ + IK \cos \alpha),$$

$$IK \sin \alpha = AI \frac{\sin(\beta - C - \frac{A}{2})}{\sin(\alpha + \beta)} \sin \alpha, \quad IK \cos \alpha = AI \frac{\sin(\beta - C - \frac{A}{2})}{\sin(\alpha + \beta)} \cos \alpha,$$

It suffices to show that

$$(2R - NQ)(NQ \sin(\alpha + \beta) - AI \sin(\beta - C - \frac{A}{2}) \sin \alpha) = (IQ + LI)(IQ \sin(\alpha + \beta) + AI \sin(\beta - C - \frac{A}{2}) \cos \alpha),$$

$$\begin{aligned} & \frac{2}{R}(IQ \sin(\alpha + \beta) + AI \sin(\beta - C - \frac{A}{2}) \cos \alpha) \\ &= 2(\sin B - \sin C) \sin(\alpha + \beta) + 4(\cos \frac{B-C}{2} - \sin \frac{A}{2}) \sin(\beta - C - \frac{A}{2}) \cos \alpha \\ &= \cos(\alpha + \beta - B) - \cos(\alpha + \beta + B) - \cos(\alpha + \beta - C) + \cos(\alpha + \beta + C) \\ & \quad + 2(\cos \frac{B-C}{2} - \sin \frac{A}{2})(\sin(\alpha + \beta - C - \frac{A}{2}) + \sin(-\alpha + \beta - C - \frac{A}{2})) \\ &= -\cos(\alpha + \beta + B - C) - \cos(\alpha + \beta) - \cos(-\alpha + \beta + B - C) - \cos(-\alpha + \beta) \\ & \quad + \cos(\alpha + \beta - B) + \cos(\alpha + \beta + C) + \cos(-\alpha + \beta + B) + \cos(-\alpha + \beta - C), \\ & \frac{2}{R}(NQ \sin(\alpha + \beta) - AI \sin(\beta - C - \frac{A}{2}) \sin \alpha) \\ &= 4 \sin \frac{A}{2} \cos \frac{B-C}{2} \sin(\alpha + \beta) - 4(\cos \frac{B-C}{2} - \sin \frac{A}{2}) \sin(\beta - C - \frac{A}{2}) \sin \alpha \\ &= 2(\cos B + \cos C) \sin(\alpha + \beta) + 2(\sin \frac{A}{2} - \cos \frac{B-C}{2}) \\ & \quad (\cos(-\alpha + \beta - C - \frac{A}{2}) - \cos(\alpha + \beta - C - \frac{A}{2})) \\ &= \sin(\alpha + \beta + B) + \sin(\alpha + \beta - B) + \sin(\alpha + \beta + C) + \sin(\alpha + \beta - C) \\ & \quad + \sin(-\alpha + \beta - C) + \sin(-\alpha + \beta + B) - \sin(-\alpha + \beta + B - C) - \sin(-\alpha + \beta) \\ & \quad - \sin(\alpha + \beta - C) - \sin(\alpha + \beta + B) + \sin(\alpha + \beta + B - C) + \sin(\alpha + \beta) \\ &= \sin(\alpha + \beta - B) + \sin(\alpha + \beta + C) + \sin(-\alpha + \beta - C) + \sin(-\alpha + \beta + B) \\ & \quad - \sin(-\alpha + \beta + B - C) - \sin(-\alpha + \beta) + \sin(\alpha + \beta + B - C) + \sin(\alpha + \beta), \\ & \cot \alpha = \frac{p-c-c \cos B}{c \sin B - r}, \quad \frac{BY}{YC} = \frac{c \sin \angle EAC}{b \sin \angle BAE} = \frac{c^2(p-b)}{b^2(p-c)}, \\ & \frac{BY}{c} = \sin B \cot \beta + \cos B = \frac{ac(p-b)}{c^2(p-b) + b^2(p-c)}, \end{aligned}$$

$$EY \sin \beta + r \cos \beta = (p - c - c \cos B) \sin \beta + (r - c \sin B) \cos \beta = \frac{c \sin B - r}{\sin \alpha} \sin(\beta - \alpha),$$

$$c \sin B - r = AI \cos \frac{B - C}{2}, \quad LI = \frac{2R \sin \beta \sin \frac{A}{2} \sin \alpha}{\cos \frac{B-C}{2} \sin(\beta - \alpha)},$$

$$\frac{p - c - c \cos B}{R} = \sin A + \sin B - \sin C - 2 \sin C \cos B = \sin B - \sin C + \sin(B - C),$$

$$\frac{c \sin B - r}{R} = 2 \sin C \sin B - (\cos A + \cos B + \cos C - 1) = \cos(C - B) - \cos B - \cos C + 1,$$

$$\cot \alpha = \frac{\sin B - \sin C + \sin(B - C)}{\cos(C - B) - \cos B - \cos C + 1},$$

$$\begin{aligned} \frac{ac(p - b)}{c^2(p - b) + b^2(p - c)} &= \frac{\sin A \sin C (\sin A + \sin C - \sin B)}{\sin^2 C (\sin A + \sin C - \sin B) + \sin^2 B (\sin A + \sin B - \sin C)} \\ &= \frac{\sin C (\sin A + \sin C - \sin B)}{\sin^2 B + \sin^2 C + \sin(C - B)(\sin C - \sin B)} = \sin B \cot \beta + \cos B, \end{aligned}$$

$$\sin C - \cos B \sin(C - B) = \sin C \sin^2 B + \sin B \cos B \cos C = \sin B \cos(B - C),$$

$$\sin C (\sin A + \sin C - \sin B) - \cos B (\sin^2 B + \sin^2 C) - \cos B \sin(C - B) (\sin C - \sin B)$$

$$= (\sin C - \sin B) \sin B \cos(B - C) + \sin C \sin B \cos C - \sin^2 B \cos B$$

$$= \sin B ((\sin C - \sin B) \cos(B - C) + \sin(B - C) \cos A),$$

$$\cot \beta = \frac{(\sin C - \sin B) \cos(B - C) + \sin(B - C) \cos A}{1 + \cos A \cos(B - C) + \sin(C - B)(\sin C - \sin B)},$$

The above expansion formulas for $\cot \alpha$, $\cot \beta$ are verified by c++ program.

$$\tan(\beta - \alpha) = \frac{\cot \beta - \cot \alpha}{1 + \cot \beta \cot \alpha} = \frac{\text{numerator}}{\text{denominator}},$$

$$\text{numerator} = ((\sin C - \sin B) \cos(B - C) + \sin(B - C) \cos A)(\cos(C - B) - \cos B - \cos C + 1)$$

$$- (\sin B - \sin C + \sin(B - C))(1 + \cos A \cos(B - C) + \sin(C - B)(\sin C - \sin B))$$

$$= 2(\sin C - \sin B) + (\sin C - \sin B)(1 - \cos B - \cos C) \cos(B - C) + \sin(B - C) \cos A(1 - \cos B - \cos C)$$

$$- \sin(B - C) - (\sin B - \sin C)(\cos A \cos(B - C) + \sin(C - B)(\sin C - \sin B))$$

$$\text{denominator} = ((\sin C - \sin B) \cos(B - C) + \sin(B - C) \cos A)(\sin B - \sin C + \sin(B - C))$$

$$+ (1 + \cos A \cos(B - C) + \sin(C - B)(\sin C - \sin B))(\cos(C - B) - \cos B - \cos C + 1)$$

=

Theorem 5 (Hardy's inequality). 1) If a_1, a_2, a_3, \dots is a sequence of non-negative real numbers, then for any real number $p > 1$, we have

$$\sum_{n \geq 1} \left(\frac{a_1 + a_2 + \dots + a_n}{n} \right)^p \leq \left(\frac{p}{p-1} \right)^p \sum_{n \geq 1} a_n^p.$$

2) Integral version: if f is a measurable function with non-negative values, then

$$\int_0^{+\infty} \left(\frac{1}{x} \int_0^x f(t) dt \right)^p \leq \int_0^{+\infty} f(x)^p dx$$

Question: what if $p \leq 1$ in the statement of Hardy's inequality? The case $p = -1$ is a classical mathematical olympiad problem.

Theorem 6 (Carleman's inequality). 1) Let a_1, a_2, a_3, \dots be a sequence of non-negative real numbers, then we have

$$\sum_{n \geq 1} (a_1 a_2 \dots a_n)^{1/n} \leq e \sum_{n \geq 1} a_n.$$

2) Integral version: if f is a measurable function with non-negative values, then

$$\int_0^{+\infty} \exp\left(\frac{1}{x} \int_0^x \log f(t) dt\right) dx \leq e \int_0^{+\infty} f(x) dx.$$

Theorem 7 (Carleson's inequality). For any convex function g with $g(0) = 0$, and for any $-1 < p < +\infty$, we have

$$\int_0^{+\infty} x^p e^{-\frac{g(x)}{x}} dx \leq e^{p+1} \int_0^{+\infty} x^p e^{-g'(x)} dx.$$

Carleman's inequality follows from the case $p = 0$.

Theorem 8 (Sobolev's inequality).

Theorem 9 (Hilbert's inequality). 1) Show that for every pair of sequences of real numbers $\{a_n\}$ and $\{b_n\}$ we have

$$\sum_{m,n=1}^{\infty} \frac{a_m b_n}{m+n} \leq \pi \sqrt{\left(\sum_{m=1}^{\infty} a_m^2\right) \left(\sum_{n=1}^{\infty} b_n^2\right)},$$

and the constant π is optimal.

2) For any nonnegative sequences $\{a_n\}$ and $\{b_n\}$ we have

$$\sum_{m,n=1}^{\infty} \frac{a_m b_n}{m+n} \leq \frac{\pi}{\sin(\frac{\pi}{p})} \left(\sum_{m=1}^{\infty} a_m^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} b_n^q\right)^{\frac{1}{q}},$$

where $p > 1, \frac{1}{p} + \frac{1}{q} = 1$, and the constant $\frac{\pi}{\sin(\frac{\pi}{p})}$ is optimal.

3) Generalization with order of denominator modified

$$\sum_{m,n=1}^{\infty} \frac{a_m b_n}{(m+n)^\tau} < \left(\frac{\pi}{\sin(\frac{\pi(q-1)}{\tau q})}\right)^\tau \|a\|_p \|b\|_q,$$

valid for $p, q > 1, \tau > 0, \frac{1}{p} + \frac{1}{q} \geq 1$ and $\tau + \frac{1}{p} + \frac{1}{q} = 2$.

4) Harder Hilbert inequality, for complex sequences $\{a_n\}$ and $\{b_n\}$ we have

$$\left| \sum_{m \neq n} \frac{a_m \bar{b}_n}{m-n} \right| \leq \pi \sqrt{\left(\sum_{m=1}^{\infty} |a_m|^2\right) \left(\sum_{n=1}^{\infty} |b_n|^2\right)}$$

Proof. 1)

$$\left(\sum_{m,n=1}^{\infty} \frac{a_m b_n}{m+n}\right)^2 \leq \left(\sum_{m,n=1}^{\infty} \frac{a_m^2}{m+n} \left(\frac{m}{n}\right)^{2\lambda}\right) \left(\sum_{m,n=1}^{\infty} \frac{b_n^2}{m+n} \left(\frac{n}{m}\right)^{2\lambda}\right),$$

$$\sum_{m,n=1}^{\infty} \frac{a_m^2}{m+n} \left(\frac{m}{n}\right)^{2\lambda} = \sum_{m=1}^{\infty} a_m^2 \sum_{n=1}^{\infty} \frac{1}{m+n} \left(\frac{m}{n}\right)^{2\lambda},$$

$$\frac{1}{m+n} \left(\frac{m}{n}\right)^{2\lambda} \leq \int_0^{\infty} \frac{1}{m+x} \frac{m^{2\lambda}}{x^{2\lambda}} dx = \int_0^{\infty} \frac{1}{(1+y)y^{2\lambda}} dy = \frac{\pi}{\sin 2\pi\lambda},$$

Choose $\lambda = \frac{1}{4}$ finishes the proof of the original L^2 Hilbert's inequality.

2) Using Holder's inequality, we have

$$\sum_{m,n=1}^{\infty} \frac{a_m b_n}{m+n} \leq \left(\sum_{m,n=1}^{\infty} \frac{a_m^p}{m+n} \left(\frac{m}{n}\right)^{p\lambda} \right)^{\frac{1}{p}} \left(\sum_{m,n=1}^{\infty} \frac{b_n^q}{m+n} \left(\frac{n}{m}\right)^{q\lambda} \right)^{\frac{1}{q}}$$

$$\sum_{m,n=1}^{\infty} \frac{a_m^p}{m+n} \left(\frac{m}{n}\right)^{p\lambda} = \sum_{m=1}^{\infty} a_m^p \sum_{n=1}^{\infty} \frac{1}{m+n} \left(\frac{m}{n}\right)^{p\lambda}, \quad \sum_{n=1}^{\infty} \frac{1}{m+n} \left(\frac{m}{n}\right)^{p\lambda} \leq \frac{\pi}{\sin p\pi\lambda},$$

$$\sum_{m,n=1}^{\infty} \frac{b_n^q}{m+n} \left(\frac{n}{m}\right)^{q\lambda} = \sum_{n=1}^{\infty} b_n^q \sum_{m=1}^{\infty} \frac{1}{m+n} \left(\frac{n}{m}\right)^{q\lambda}, \quad \sum_{m=1}^{\infty} \frac{1}{m+n} \left(\frac{n}{m}\right)^{q\lambda} \leq \frac{\pi}{\sin q\pi\lambda},$$

$$\text{minimize } \left(\frac{\pi}{\sin p\pi\lambda}\right)^{\frac{1}{p}} \left(\frac{\pi}{\sin q\pi\lambda}\right)^{\frac{1}{q}}, \quad F(\lambda) = \frac{1}{p} \log \sin p\pi\lambda + \frac{1}{q} \log \sin q\pi\lambda,$$

$$F'(\lambda) = \cot p\pi\lambda + \cot q\pi\lambda = 0, \quad \lambda = \frac{1}{p+q}, \quad \frac{p}{p+q} = \frac{1}{q}, \quad \frac{q}{p+q} = \frac{1}{p},$$

$$\sum_{m,n=1}^{\infty} \frac{a_m b_n}{m+n} \leq \frac{\pi}{\sin(\frac{\pi}{p})} \left(\sum_{m=1}^{\infty} a_m^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} b_n^q\right)^{\frac{1}{q}},$$

Observation: let $f(z) = \sum_{n>0} a_n e^{-nz}$, then

$$\sum_{m,n=1}^{\infty} \frac{a_m a_n}{m+n} = \int_0^{+\infty} |f(t)|^2 dt, \quad 2\pi \sum_{n=1}^{\infty} a_n^2 = \int_0^{2\pi i} |f(t)|^2 dt,$$

How to convert the inequality to terms inside $\Omega = \{z \in \mathbb{C}, \Re(z) \geq 0, 0 \leq \Im(z) \leq 2\pi\}$?

4) Using Toeplitz's method, since $\int_0^{2\pi} (t-\pi)e^{int} dt = \frac{2\pi}{in}$, we have

$$\sum_{m,n=1}^{\infty} \frac{a_m \bar{b}_n}{m-n} = \frac{i}{2\pi} \int_0^{2\pi} (t-\pi) \left(\sum_{m \geq 1} a_m e^{imt}\right) \left(\sum_{n \geq 1} \bar{b}_n e^{-int}\right) dt,$$

$$|RHS| \leq \frac{\|t-\pi\|_{\infty}}{2\pi} \left| \int_0^{2\pi} \left(\sum_{m \geq 1} a_m e^{imt}\right) \left(\sum_{n \geq 1} \bar{b}_n e^{-int}\right) dt \right| \leq \pi \|a\|_2 \|b\|_2,$$

□

Question 4. 1) Guaranteed positivity: show that for any real numbers a_1, a_2, \dots, a_n and positive $\lambda_1, \lambda_2, \dots, \lambda_n$ one has

$$\sum_{i,j=1}^n \frac{a_i a_j}{i+j} \geq 0, \quad \sum_{i,j=1}^n \frac{a_i a_j}{\lambda_i + \lambda_j} \geq 0,$$

2) Show that if the complex array $\{a_{jk}\}$ satisfies the bound

$$\left| \sum_{j,k} a_{jk} x_j y_k \right| \leq M \|x\|_2 \|y\|_2,$$

then one also has the bound

$$\left| \sum_{j,k} a_{jk} h_{jk} x_j y_k \right| \leq \alpha \beta M \|x\|_2 \|y\|_2,$$

provided that the factors h_{jk} have an integral representation of the form

$$h_{jk} = \int_D f_j(x) g_k(x) dx$$

and for all j, k one has the bounds

$$\int_D |f_j(x)|^2 dx \leq \alpha^2, \quad \int_D |g_k(x)|^2 dx \leq \beta^2,$$

3) Show that for every pair of sequences of real numbers $\{a_n\}$ and $\{b_n\}$ one has

$$\sum_{m,n=1}^{\infty} \frac{a_m b_n}{\max(m,n)} \leq 4 \sqrt{\left(\sum_{m=1}^{\infty} a_m^2 \right) \left(\sum_{n=1}^{\infty} b_n^2 \right)},$$

and the constant 4 is optimal.

4) Carlson's inequality:

$$\left(\sum_{k=1}^n a_k \right)^4 \leq \pi^2 \left(\sum_{k=1}^n a_k^2 \right) \left(\sum_{k=1}^n k^2 a_k^2 \right)$$

5) Hilbert's inequality via the Toeplitz method: the Fourier coefficients of $t - \pi, 0 \leq t \leq 2\pi$ are

$$\int_0^{2\pi} (t - \pi) e^{int} dt = \frac{(t - \pi) e^{int}}{in} \Big|_0^{2\pi} - \frac{1}{in} \int_0^{2\pi} e^{int} dt = \frac{2\pi}{in},$$

so for real $a_k, b_k, k \geq 1$ one has

$$\begin{aligned} \sum_{m,n=1}^{\infty} \frac{a_m b_n}{m+n} &= \frac{i}{2\pi} \int_0^{2\pi} (t - \pi) \left(\sum_{m \geq 1} a_m e^{imt} \right) \left(\sum_{n \geq 1} b_n e^{int} \right) dt, \\ RHS &\leq \frac{\|t - \pi\|_{\infty}}{2\pi} \left| \int_0^{2\pi} \left(\sum_{m \geq 1} a_m e^{imt} \right) \left(\sum_{n \geq 1} b_n e^{int} \right) dt \right| \leq \pi \|a\|_2 \|b\|_2, \end{aligned}$$

the last step used the following fact: $\tilde{a}(t) = \sum_{m \geq 1} a_m e^{imt}, \tilde{b}(t) = \sum_{n \geq 1} b_n e^{int}$,

$$\left| \int_0^{2\pi} \tilde{a}(t) \tilde{b}(t) dt \right| \leq \|\tilde{a}\|_2 \|\tilde{b}\|_2 = 2\pi \|a\|_2 \|b\|_2,$$

6) Generalization: i) for any $\lambda \in [0, 1]$, we have

$$\left| \sum_{m,n>0} \frac{a_m \bar{b}_n}{m - n - \lambda} \right| \leq \frac{\pi}{\sin(\lambda\pi)} \|a\|_2 \|b\|_2,$$

and the constant $\frac{\pi}{\sin(\lambda\pi)}$ is optimal.

ii) If $\lambda \geq \frac{1}{2}$, then

$$\sum_{j,l \geq 0} \frac{a_j \bar{b}_l}{j + l + \lambda} \leq \pi \|a\|_2 \|b\|_2,$$

If $0 < \lambda < \frac{1}{2}$, then

$$\sum_{j,l \geq 0} \frac{a_j \bar{b}_l}{j + l + \lambda} \leq \frac{\pi}{\sin(\lambda\pi)} \|a\|_2 \|b\|_2,$$

Proof. 6) i) Let $F(t) = \frac{i}{e^{-2\pi i\lambda} - 1} e^{-i\lambda t}$, then $\int_0^{2\pi} F e^{int} dt = \frac{1}{n-\lambda}$, $\|F\|_\infty = \frac{1}{2\sin(\lambda\pi)}$. Using Toeplitz method, we have

$$\begin{aligned} & \left| \sum_{m,n>0} \frac{a_m \bar{b}_n}{m-n-\lambda} \right| = \left| \int_0^{2\pi} F \left(\sum_{m \geq 1} a_m e^{imt} \right) \left(\sum_{n \geq 1} \bar{b}_n e^{-int} \right) dt \right| \\ & \leq \|F\|_\infty 2\pi \|a\|_2 \|b\|_2 = \frac{\pi}{\sin(\lambda\pi)} \|a\|_2 \|b\|_2, \end{aligned}$$

□

Theorem 10 (Pólya's random walk theorem). A random walk is said to be recurrent if it returns to its initial position with probability one. A random walk which is not recurrent is called transient. Pólya's classical result says: the simple random walk on \mathbb{Z}^d is recurrent in dimensions $d = 1, 2$ and transient in dimensions $d \geq 3$.

Proof.

□

Question 5 (Spectrum of one dimensional quantum harmonic oscillator). Find the values of λ such that

$$-u'' + x^2 u = \lambda u, \quad x \in \mathbb{R}, u \neq 0,$$

we may assume that u is a Schwartz function.

Proof. All the eigenvalues are $\lambda_n = 2n + 1, n \in \mathbb{N}$, with eigenfunctions $u_n = e^{-\frac{x^2}{2}} H_n(x)$ where H_n is the degree n Hermite polynomial. Substitute $u = e^{-\frac{x^2}{2}} \tilde{u}$, we have

$$\begin{aligned} -u'' + x^2 u &= e^{-\frac{x^2}{2}} (-\tilde{u}'' + 2x\tilde{u}' - (x^2 - 1)\tilde{u} + x^2\tilde{u}), \\ -\tilde{u}'' + 2x\tilde{u}' &= (\lambda - 1)\tilde{u}, \end{aligned}$$

First, we show that all eigenvalues are nonnegative. It follows by

$$\langle -u'' + x^2 u, u \rangle = \|u'\|_2^2 + \|xu\|_2^2 \geq 0, \quad \lambda \geq 0,$$

- 1) \tilde{u} is a nonzero constant, then it is a valid solution with eigenvalue $\lambda = 1$.
- 2) \tilde{u} is not a constant. Let $v = \partial_x \tilde{u}$, then it satisfies

$$\partial_x(-\tilde{u}'' + 2x\tilde{u}') = -v'' + 2xv' + 2v = (\lambda - 1)v, \quad -v'' + 2xv' = (\lambda - 3)v,$$

so v is an eigenfunction with eigenvalue $\lambda - 2$.

We use an induction on $[\lambda]$ to find all eigenvalues and eigenfunctions. i) If $[\lambda] \leq 1$ and \tilde{u} is not a constant, then $v = \partial_x \tilde{u}$ is an eigenfunction with eigenvalue $\lambda - 2 < 0$, contradiction! So if $[\lambda] \leq 1$, \tilde{u} must be a constant and $\lambda = 1$. ii) If $[\lambda] \geq 2$, then \tilde{u} mustn't be a constant. $v = \partial_x \tilde{u}$ is an eigenfunction with eigenvalue $\lambda - 2$. Since $[\lambda - 2] = [\lambda] - 2$, v is determined by induction. Then \tilde{u} is determined by equation

$$(\lambda - 1)\tilde{u} = -v' + 2xv,$$

Variation of parameters: assume $u = e^{Q(x)} \tilde{u}$, then

$$u'' = e^{Q(x)} (\tilde{u}'' + 2Q'(x)\tilde{u}' + (Q''(x) + Q'(x)^2)\tilde{u})$$

to the leading order let $Q'(x)^2 = x^2$, we get $Q = -\frac{x^2}{2} + C$.

□

Question 6 (Spectrum of higher dimensional quantum harmonic oscillator). Find the values of λ such that

$$\mathcal{L}_0 u = -\Delta u + |x|^2 u = \lambda u, \quad x \in \mathbb{R}^n, u \neq 0,$$

we may assume that u is a Schwartz function.

Proof. Variation of parameters: $u = f\tilde{u}$, $f = e^{-\frac{|x|^2}{2}}$, we have

$$\Delta u = f\Delta\tilde{u} + 2\nabla f \cdot \nabla\tilde{u} + \tilde{u}\Delta f, \quad \nabla f = -\bar{x}f, \quad \Delta f = (|x|^2 - n)f,$$

$$\mathcal{L}\tilde{u} = -\Delta\tilde{u} + 2\bar{x} \cdot \nabla\tilde{u} = (\lambda - n)\tilde{u},$$

Energy estimate shows that all eigenvalues are nonnegative:

$$\langle -\Delta u + |x|^2 u, u \rangle = \|\nabla u\|_2^2 + \|\bar{x}u\|_2^2 \geq 0, \quad \lambda \geq 0,$$

1) \tilde{u} is a nonzero constant, then it is a valid solution with eigenvalue $\lambda = n$.

2) \tilde{u} is not a constant. Let $v = \partial_x \tilde{u}$, then it satisfies

$$\partial_x(-\Delta\tilde{u} + 2\bar{x} \cdot \nabla\tilde{u}) = -\Delta v + 2\bar{x} \cdot \nabla v + 2v = (\lambda - n)v, \quad \mathcal{L}v = (\lambda - n - 2)v,$$

so v is an eigenfunction with eigenvalue $\lambda - 2$.

□

References

- [1] Stanley, Enumerative combinatorics.